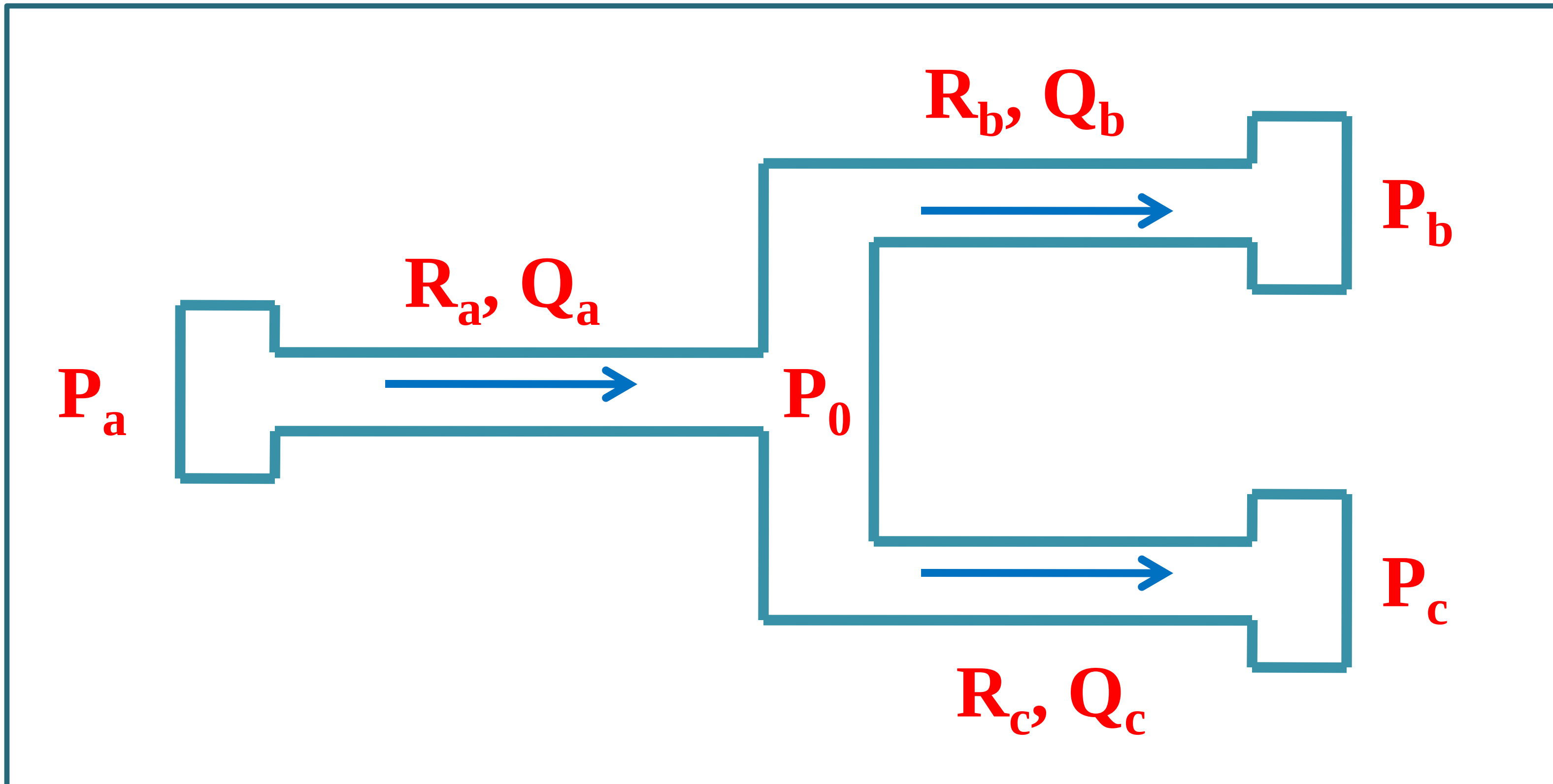
Solving Linear Systems

Jake Blanchard
University of Wisconsin - Madison
Spring 2008

Case Study

- Solve for flow rates and pressures in a piping network
- 3 equations for flow in a leg
- 1 conservation equation
- Unknowns are 3 flow rates and a pressure

Schematic



The Case

- p_a, p_b, p_c, R_a, R_b, R_c are known
- Solve for p_0, Q_a, Q_b, Q_c

$$Q_a R_a = p_a - p_0$$

$$Q_b R_b = p_0 - p_b$$

$$Q_c R_c = p_0 - p_c$$

$$Q_a = Q_b + Q_c$$

Solution Techniques

- Convert to matrix equation
- Solve using \ operator

Example

$$x + 2y + z = 1$$

$$x + y - z = 0$$

$$2x - y + 2z = 1$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & -1 \\ 2 & -1 & 2 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 1 \\ 0 \\ 1 \end{Bmatrix}$$

Terminology

- **Ax=b**
- A=matrix
- x=solution vector
- b=loading vector (right-hand-side of equation)
- A and b known, solve for x

Using Matlab

- **A=[1 2 1; 1 1 -1; 2 -1 2];**
- **b=[1; 0; 1];**
- **x=A\b**

Practice

- Solve this problem (previous slide)

Practice

- Revise that problem to this:

$$x + 2y + z = 1$$

$$x - z = 0$$

$$2x - y = 1$$

Uniqueness of Solutions

- For a system of N equations in N unknowns, a unique solution exists only if the determinant of A is not zero
- In Matlab, det(A) will give the determinant of A

What are the determinants of these matrices?

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 2 & 2 & 0 \end{bmatrix}$$

Solve this matrix equation

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 2 & 2 & 0 \end{bmatrix} \begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \begin{Bmatrix} 1 \\ -1 \\ 0 \end{Bmatrix}$$

Condition Numbers

- Some matrices are "almost singular"
- The condition number is a measure of this characteristic
- If the condition number is large, errors in solving corresponding matrix equations will tend to be large
- The command is **cond(A)**

What are the condition numbers of these matrices?

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 2 & -1 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 2 & 2 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & -1 \\ 2 & 2 & 10^{-5} \end{bmatrix}$$

The Hilbert matrix is defined below…

- Find the condition number of the Hilbert matrix for N=3, 5, and 7

$$H_{ij} = \frac{1}{i+j-1}$$

$$H = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}$$

$$H = \begin{bmatrix} 1 & 1/2 & 1/3 \\ 1/2 & 1/3 & 1/4 \\ 1/3 & 1/4 & 1/5 \end{bmatrix}$$

Code for creating Hilbert matrix

```
Hsize=7
for i=1:Hsize
    for j=1:Hsize
        hilb(i,j)=1/(i+j-1);
    end
end
disp(hilb)
```

Practice

- **Solve pipe flow problem**
  - **P_a=12,500 lb/ft^2**
  - **P_b=3,500 lb/ft^2**
  - **P_c=2,000 lb/ft^2**
  - **R_a=R_b=R_c=6,000 lb-s/ft^5**

$$Q_a R_a = p_a - p_0$$

$$Q_b R_b = p_0 - p_b$$

$$Q_c R_c = p_0 - p_c$$

$$Q_a = Q_b + Q_c$$

My Matrix

$$
\begin{array}{cccccc}
R_a Q_a & +0 & +0 & +p_0 & = & p_a \\
0 & +R_b Q_b & +0 & -p_0 & = & -p_b \\
0 & +0 & +R_c Q_c & -p_0 & = & -p_c \\
Q_a & -Q_b & -Q_c & +0 & = & 0
\end{array}
$$

$$
\begin{bmatrix} R_a & 0 & 0 & 1 \\ 0 & R_b & 0 & -1 \\ 0 & 0 & R_c & -1 \\ 1 & -1 & -1 & 0 \end{bmatrix}
\begin{Bmatrix} Q_a \\ Q_b \\ Q_c \\ p_0 \end{Bmatrix}
=
\begin{Bmatrix} p_a \\ -p_b \\ -p_c \\ 0 \end{Bmatrix}
$$

Questions