Numerical Integration

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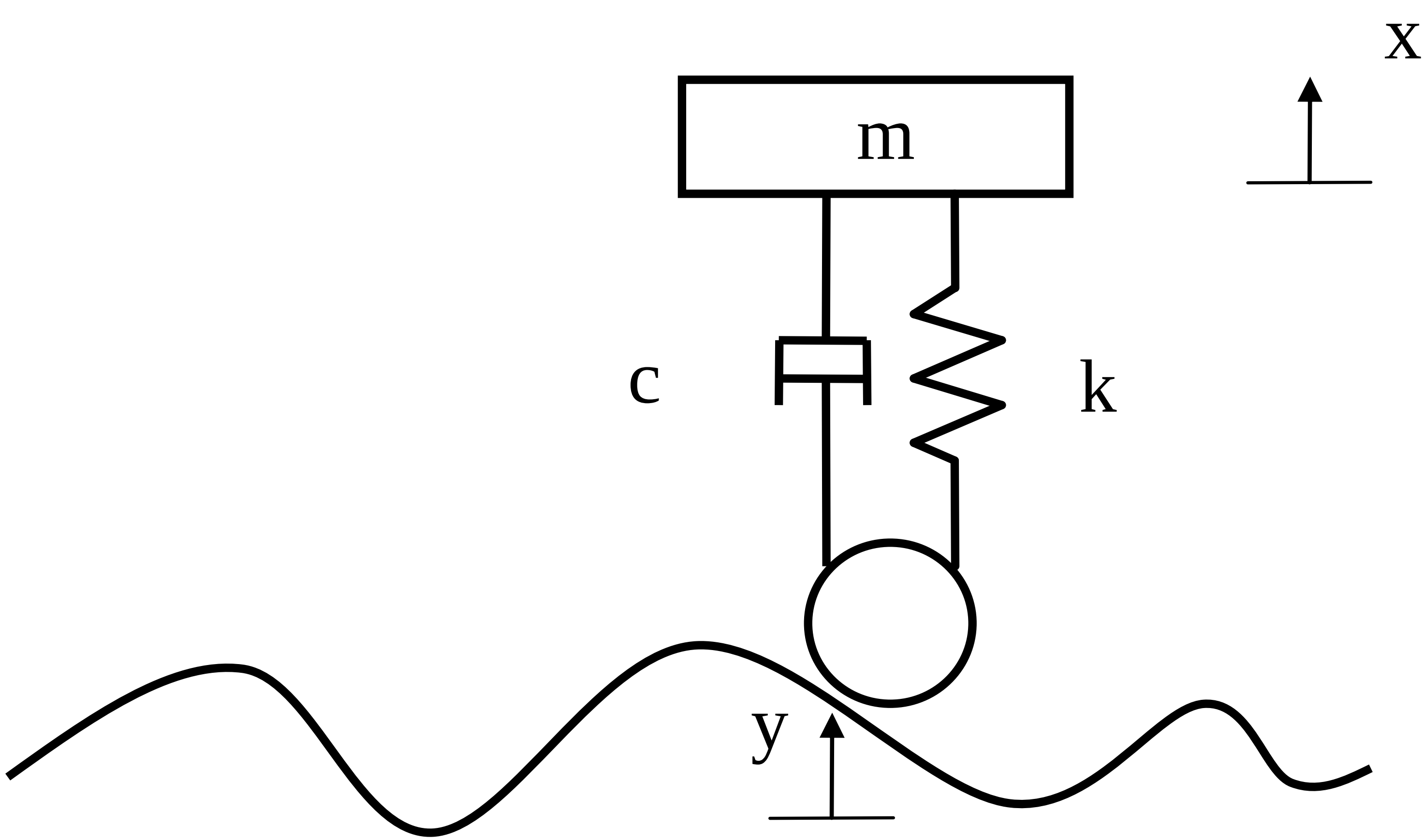
Case Study



The Case

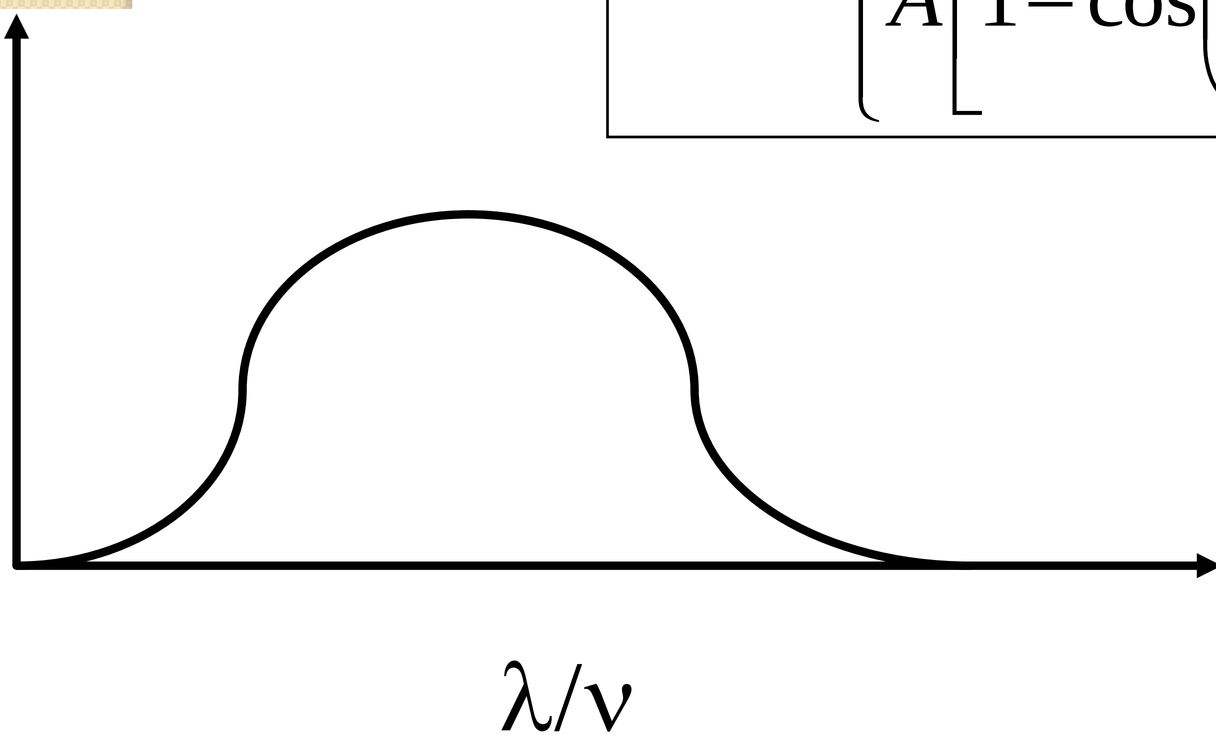
- Model as a differential equation (F=ma)
- Then convert to integral

$$x(t) - y(t) = -\frac{1}{\omega_n}\int_0^t \frac{d^2 y}{dt^2} h(t-s)\,ds$$

$$h(t) = \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin\left(\sqrt{1-\zeta^2}\,\omega_n t\right)$$

The Case

$$y(t) = \begin{cases} 0 & t > \dfrac{\lambda}{V} \\ A\left[1 - \cos\left(2\pi \dfrac{Vt}{\lambda}\right)\right] & t < \dfrac{\lambda}{V} \end{cases}$$



The Case

- Note that since y=0 for t>λ/V, then we can write, for t> λ/V

$$x(t) - y(t) = -\frac{1}{\omega_n}\int_0^{\lambda/V} \frac{d^2 y}{dt^2} h(t-s)ds$$

Simplifying…

- Look at end of bump ($t=\lambda/V$) and ignoring damping

$$x = -\frac{A}{\omega_n}\left(\frac{2\pi V}{\lambda}\right)^2 \int_0^{\lambda/V} \cos\left(\frac{2\pi V\xi}{\lambda}\right) \sin\left(\omega_n\left[\frac{\lambda}{V} - \xi\right]\right) d\xi$$

- k=60,000 N/m
- m=900 kg
- λ=4 m A=0.04 m V= 75 mph

Simplifying…

$$x = 13.6\int_0^{0.12} \cos(53\,s)\sin[8(0.12 - s)]\,ds$$

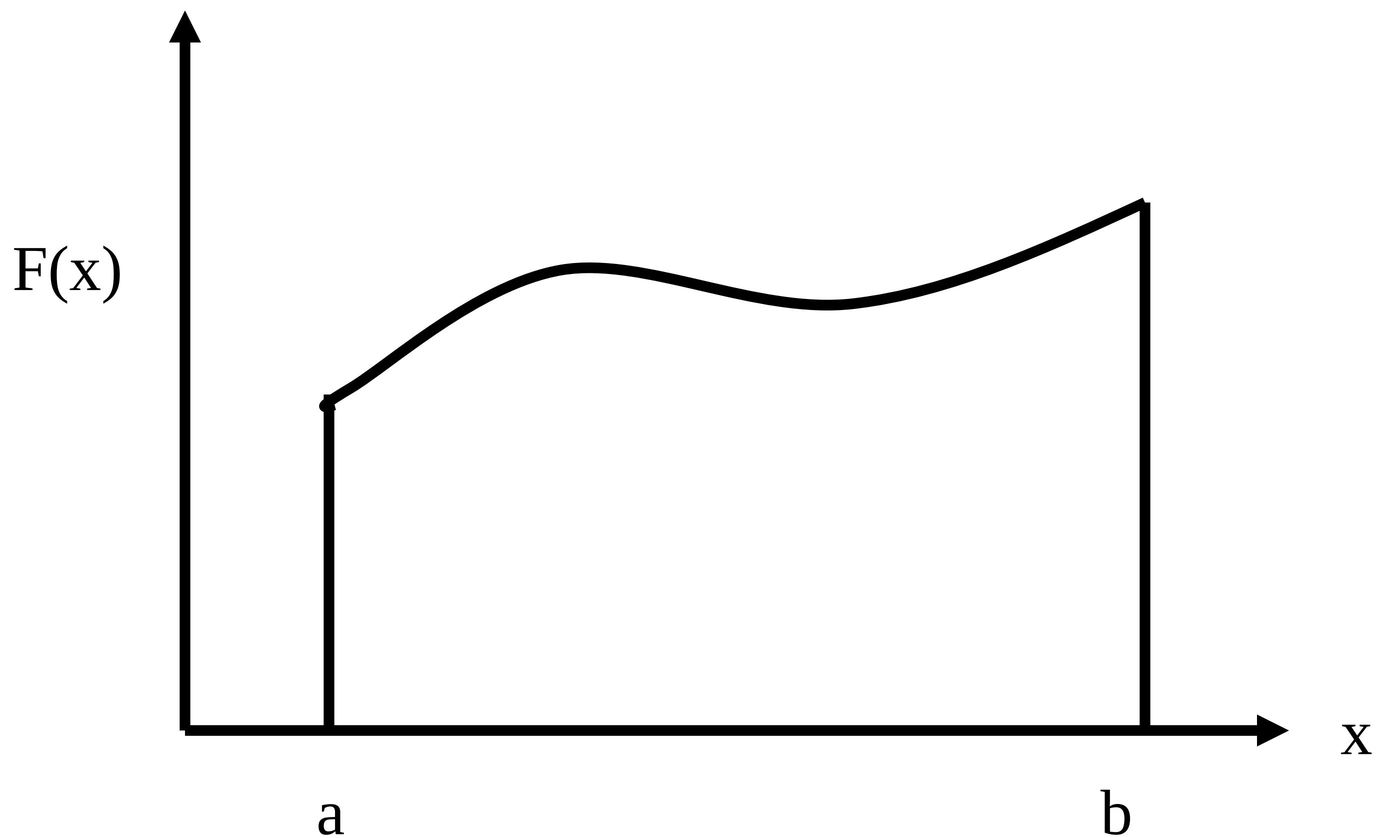
What is numerical integration?

- We seek a numerical approximation to

$$I = \int_a^b f(x)dx$$

- This is equivalent to finding the area under a curve

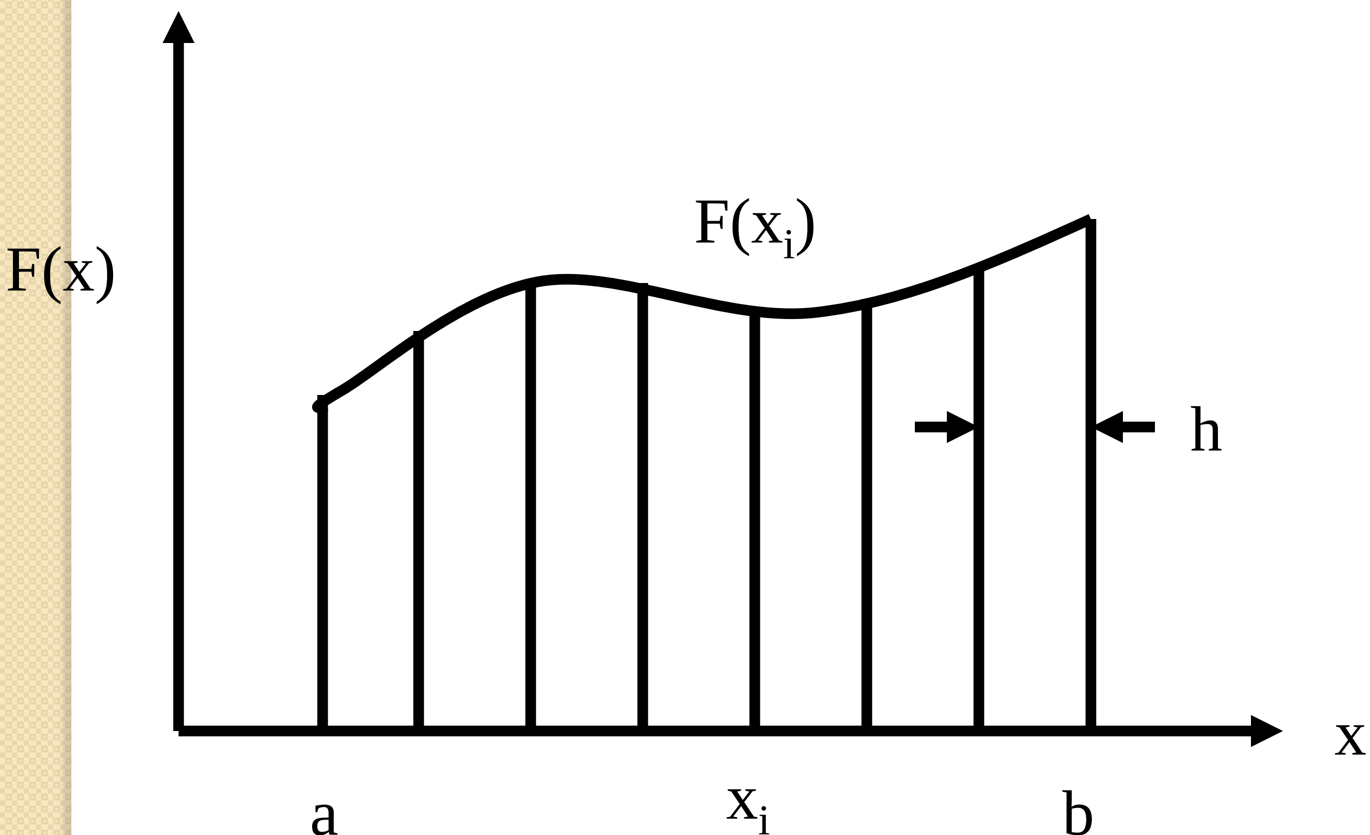
Schematic



General Approach

- Divide interval $(a<x<b)$ up into small pieces
- Evaluate f(x) at discrete points
- Approximate integral as sum off approximate areas of pieces

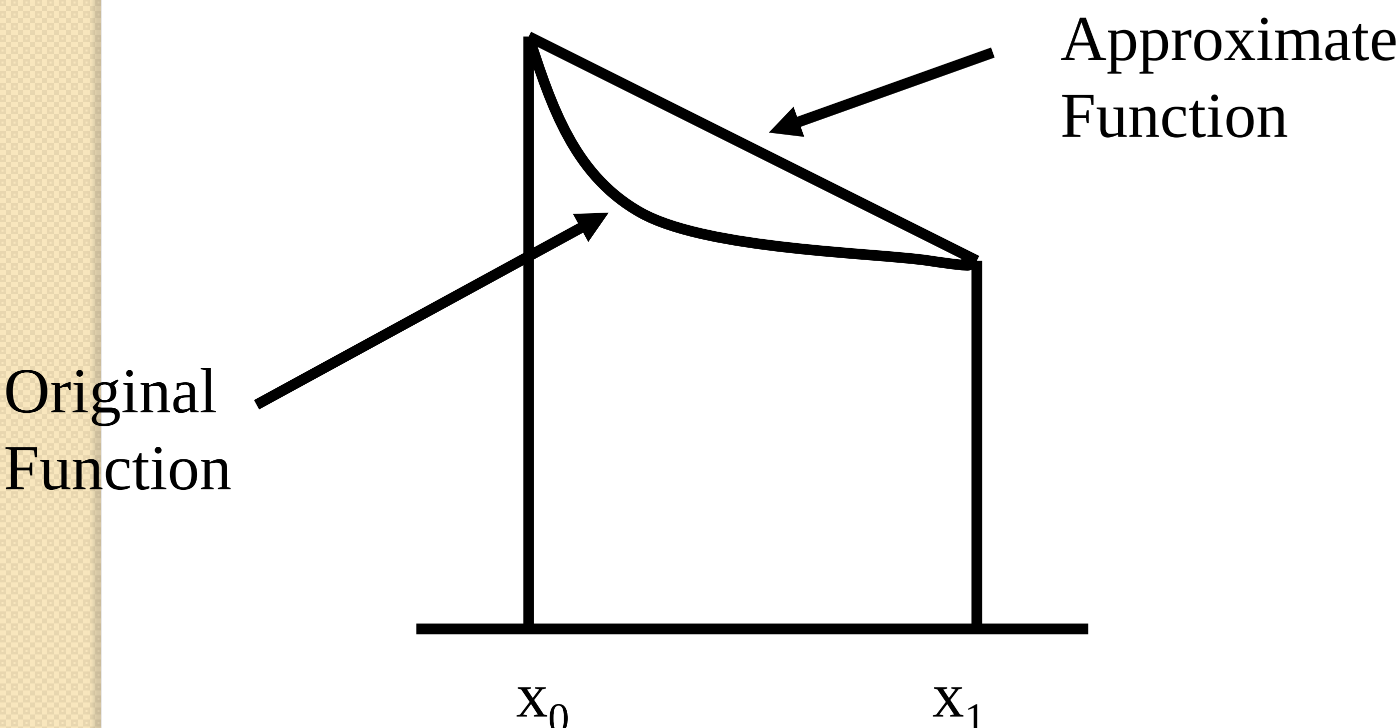
Schematic



Trapezoidal Rule

- This is the simplest rule
- Connect two points at top to create trapezoid
- Area of trapezoid is width times average height

Trapezoidal Rule



Trapezoidal Rule

- Approximate Area=0.5*h*[$F(x_0)+F(x_1)$]
- Now we just add up all the little areas to get the full area
- For 2 divisions, we get
- A= 0.5*h*{[$F(x_0)+F(x_1)$]+
  [$F(x_1)+F(x_2)$]}

Trapezoidal Rule

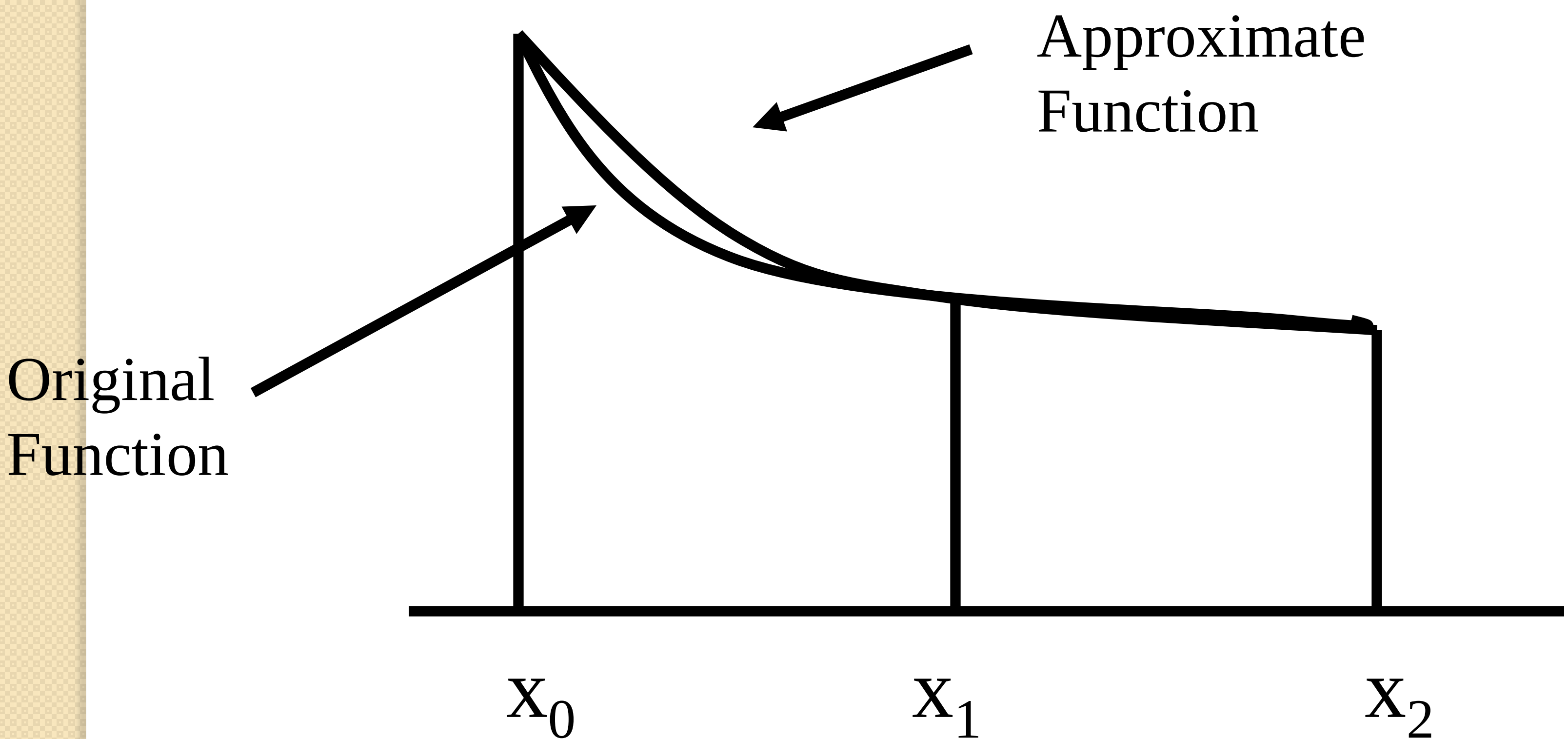
- Or: $A = h*[0.5*F(x_0)+F(x_1)+0.5*F(x_2)]$
- The composite rule, then, is that we add up half the first and last points along with all the interior points

$$I \approx h\left(\frac{f_0}{2} + f_1 + f_2 + \cdots + f_{N-2} + f_{N-1} + \frac{f_N}{2}\right)$$

Simpson's Rule

- Similar to trapezoidal rule
- Use pairs of divisions
- Fit to parabola at top

Simpson's Rule



Simpson's Rule

- The composite rule

$$I \approx \frac{h}{3}\begin{pmatrix} f_0 + 4f_1 + 2f_2 + 4f_3 \cdots + \\ 4f_{N-3} + 2f_{N-2} + 4f_{N-1} + f_N \end{pmatrix}$$

Matlab

- With Matlab, you can just use the QUADL function that is built into the program
- This will be both more accurate and faster than Excel
- It uses an adaptive Simpson's rule

Matlab

```
a=0;
b=1;
integral=quadl('sin',a,b)
integral=quadl(@sin,a,b)
```

Practice

- The length of the supporting cable of a suspension bridge is given by the integral below.
- Solve this for a=60 m and h=15 m, where a is the half-length of the bridge and h is the tower height

$$L = 2\int_0^a \sqrt{1+\frac{4h^2x^2}{a^4}}dx$$

Practice

- The electric field due to a charged circular disk at a distance z along the disk axis is given below.
- Find E at z=5 cm for R=6 cm, σ=300 μC/m^2

$$E = \frac{\sigma z}{4\varepsilon_0} \int_0^R \frac{2r dr}{\left(z^2 + r^2\right)^{1.5}}$$

$$\varepsilon_0 = 8.85 \times 10^{-12} \quad C^2 / N - m^2$$

Practice – car suspension

$$x = 13.6\int_0^{0.119} \cos(53\, s)\sin[8(0.119 - s)]ds$$

Questions