Parabolic PDE's in Matlab

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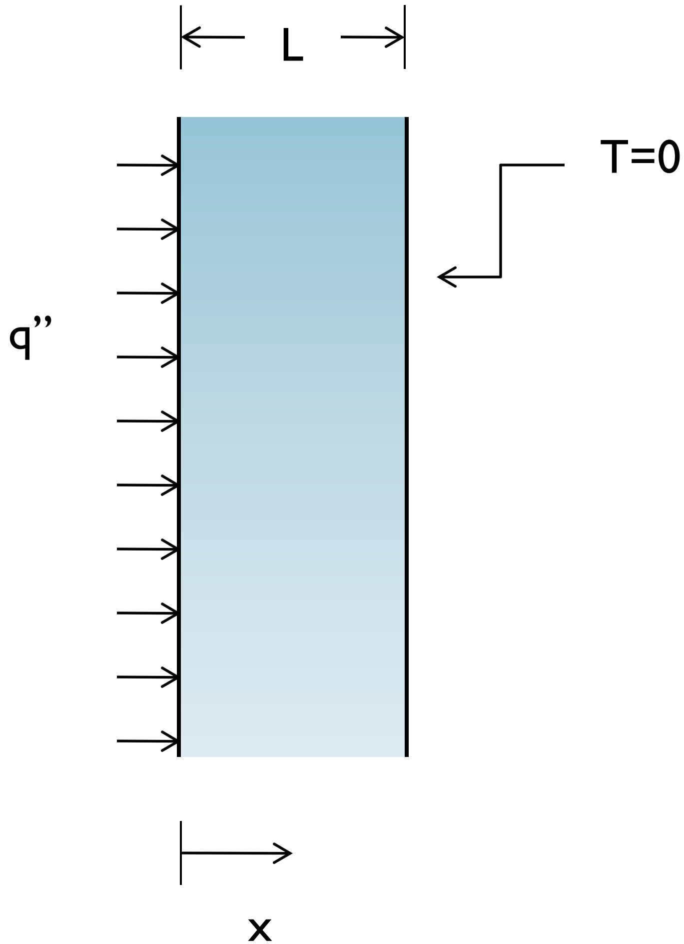
Introduction

- Parabolic partial differential equations are encountered in many scientific applications
- Think of these as a time-dependent problem in one spatial dimension
- Matlab's **pdepe** command can solve these

Model Problem



$$\rho c_p \frac{\partial T}{\partial x} = k \frac{\partial^2 T}{\partial x^2}$$

$$T(x,0) = 0$$

$$-k \left. \frac{\partial T}{\partial x} \right|_{x=0} = q''$$

$$T(L,t) = 0$$

pdepe Solves the Following

m=0 for Cartesian, 1 for cylindrical, 2 for spherical

$$c\left(x,t,u,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t}=x^{-m}\frac{\partial}{\partial x}\left(x^{m}f\left(x,t,u,\frac{\partial u}{\partial x}\right)\right)+s\left(x,t,u,\frac{\partial u}{\partial x}\right)$$

$$u(x,t_0)=u_0(x)$$

Initial Conditions

$$p(x,t,u)+q(x,t)f\left(x,t,u,\frac{\partial u}{\partial x}\right)=0$$

Boundary Conditions – one at each boundary

pdepe Solves the Following

$$c\left(x,t,u,\frac{\partial u}{\partial x}\right)\frac{\partial u}{\partial t} = x^{-m}\frac{\partial}{\partial x}\left(x^{m} f\left(x,t,u,\frac{\partial u}{\partial x}\right)\right) + s\left(x,t,u,\frac{\partial u}{\partial x}\right)$$

$$\rho c_p \frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$$

$$c = \rho c_p$$

$$f = k\frac{\partial T}{\partial x}$$

$$s = 0$$

Differential Equations

```
function [c,f,s] = pdex1pde(x,t,u,DuDx)
global rho cp k
c = rho*cp;
f = k*DuDx;
s = 0;
```

Initial Conditions

```
function u0 = pdex1ic(x)
u0 = 0;
```

pdepe Solves the Following

$$-k\left.\frac{\partial T}{\partial x}\right|_{x=0} = q''$$

$$p(x,t,u)+q(x,t)f\left(x,t,u,\frac{\partial u}{\partial x}\right)=0$$

or

$$q''+k\left.\frac{\partial T}{\partial x}\right|_{x=0} = 0$$

$x = 0$

$p = q''$

$q = 1$

remember

$$f = k\frac{\partial T}{\partial x}$$

$x = L$

$p = T = ur$

$q = 0$

$$T(L,t) = 0$$

Boundary Conditions

```
function [pl,ql,pr,qr] = pdex1bc(xl,ul,xr,ur,t)
global q
pl = q;
ql = 1;
pr = ur;
qr = 0;
```

At left edge, q+k*dT/dx=0

At right edge, ur=0

Calling the solver

```
tend=10
m = 0;
x = linspace(0,L,200);
t = linspace(0,tend,50);
sol =
  pdepe(m,@pdex1pde,@pdex1ic,@pdex1bc,x,t);
```

200 spatial mesh points

50 time steps from t=0 to tend

Postprocessing

```
Temperature = sol(:,:,1);
figure, plot(x,Temperature(end,:))

figure, plot(t,Temperature(:,1))
```

Full Code

```
function parabolic
global rho cp k
global q
L=0.1      %m
k=200      %W/m-K
rho=10000 %kg/m^3
cp=500    %J/kg-K
q=1e6   %W/m^2
tend=10 %seconds

m = 0;
x = linspace(0,L,200);
t = linspace(0,tend,50);

sol = pdepe(m,@pdex1pde,@pdex1ic,@pdex1bc,x,t);
Temperature = sol(:,:,1);
 figure, plot(x,Temperature(end,:))
```

```
function [c,f,s] = pdex1pde(x,t,u,DuDx)
global rho cp k
c = rho*cp;
f = k*DuDx;
s = 0;

function u0 = pdex1ic(x)
u0 = 0;

function [pl,ql,pr,qr] =
    pdex1bc(xl,ul,xr,ur,t)
global q
pl = q;
ql = 1;
pr = ur;
qr = 0;
```

A Second Problem

- Suppose we want convection at x=L
- That is

$$p(x,t,u)+q(x,t)f\left(x,t,u,\frac{\partial u}{\partial x}\right)=0$$

$$-k\frac{dT}{dx}=h(T-T_{bulk})$$

$$or$$

$$hT-hT_{bulk}+k\frac{dT}{dx}=0$$

$$x=L$$

$$p=T=h(ur-T_{bulk})$$

$$q=1$$

Altered Code

```
function u0 = pdex1ic(x)
global q hcoef Tbulk
u0 = Tbulk;

function [pl,ql,pr,qr] = pdex1bc(xl,ul,xr,ur,t)
global q hcoef Tbulk
pl = q;
ql = 1;
pr = hcoef*(ur-Tbulk);
qr = 1;
```

Download Scripts

http://blanchard.ep.wisc.edu/